АСТРОФИЗИКА

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DYNAMICAL EQUATIONS OF A SUPERFLUID IN CURVED SPACE-TIME AND CATTANEO'S PROJECTION METHOD

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The absolute tensorial equations describing the dynamics of a superfluid in General relativity have been brought into the same form as the corresponding classical equations written in 3-vector forms. The expressions of the various forces acting on an element of superfluid are explicitly displayed. In the Newtonian limit, these equations give the classical equations of motion of a superfluid in Galilean space-time.

Key words: *superfluid:dynamical equations - General relativity*

1. *Introduction*. The covariant equation describing the dynamics of superfluids in the presence of vortices may be written in the following form [1]

$$\eta^{ijkl}\nabla_{[k}\, hu_{l]} = S^{ij} \tag{1}$$

where ∇ denotes the operator of covariant derivation, u^i $(i=1, 2, 3, 4)$ the 4-velocity ($u_i u^i = -c^2$) of the element of superfluid and h the specific enthalpy, i.e. enthalpy per particle. The antisymmetric tensor S^{ij} embodies the properties of the system of vortices and is defined by

$$S^{ij} = -w\varepsilon^{ij}(L), \tag{2a}$$

$$\varepsilon^{ij}(L) = -u^i(L)w^j(L) + u^j(L)w^i(L), \tag{2b}$$

where $u^i(L)$ and $w^i(L)$ are respectively the 4-velocity of the vortex and the unit spacelike vector defining the direction of the vortex. The scalar w

$$w^2 = w_{ij}w^{ij}\,, \quad w_{ij} = 2\nabla_{[i}\, hu_{j]} \tag{3}$$

is proportional to the proper density of vortices. The permutation symbol η is given by

$$\eta_{ijkl} = \sqrt{-g}\, \epsilon_{ijkl}\,, \quad \eta^{ijkl} = -\frac{1}{\sqrt{-g}}\, \epsilon_{ijkl}\,. \tag{4}$$

Let us note that Eqs.(1) and (2) may also be derived from the Klein-Gordon equation describing the wave function of the superfluid condensate [2].

Substitution of (2) in (1) yields the following form for the dynamical equation of rotating superfluids

$$\nabla_{[r} hu_{s]} = S_{rs} = -\frac{1}{2} w \eta_{rsij} u^i(L) w^j(L). \tag{5}$$

As it is well known, a difficulty inherent to the theory of general relativity is the lack of a unique mathematical representation of the physically measurable quantities in terms of the corresponding absolute tensorial quantities; accordingly the physical interpretation of Eq.(5) is not immediate, it necessitates the specification of the relations connecting these two sets of quantities.

In recent notes [3,4] the physical interpretation of the covariant London equation was developed by applying Cattaneo's projection method [5]. This method has a purely tensorial character and absolute tensorial equations are transformed into a form similar to that of classical physics with added terms representing the influence of the gravitational field. This result is attained by the introduction, by means of projection operators suitably defined, of various "standard" quantities relative to the chosen system of reference S, quantities which transform according to the classical tensorial law on changes of coordinates internal to the system of reference S, and by adopting an operator of transverse derivation, partial or covariant, with respect to the x^4-lines.

The purpose of this note is to present an alternate formulation of the basic dynamical equations of superfluidity in curved space-time adopting Cattaneo's method. From the natural projections of the absolute tensor Eq.(1) we shall derive a set of two standard relative equations: (i) the standard equation of motion with respect to the chosen system of reference S, associated to the physically admissible coordinates $\{x^i\}$, displaying explicitly the various forces acting on the superfluid, (ii) the standard equation describing the standard 3-velocity field with the various physical sources which make the curl of the standard velocity different from zero.

We shall regard this note as a continuation of our previous paper [3] (henceforth referred as paper I), we thus avoid spending space in recalling the definitions and properties of: (i) a system of reference S and the natural projections of a tensor relative to S, (ii) the transverse partial and covariant derivation operators, (iii) the standard relative quantities which, for the problem under consideration, permit the introduction of the familiar language of physics. We shall suppose that the reader has paper I at hand; references of the form [I-2.3] are to be understood as equation (2.3) of paper I.

2. *Standard relative dynamical equations for superfluids.* Let V_4 be the domain of space-time occupied by the superfluid, (x^i), $i=1, 2, 3, 4$ with $x^4 = ct$ a physically admissible coordinate system and

$$ds^2 = g_{ij} dx^i dx^j \quad (g_{44} < 0,\ g_{\alpha\beta} dx^\alpha dx^\beta > 0) \quad (\alpha, \beta = 1, 2, 3) \tag{6}$$

the metric form of signature +2 with the corresponding condition of the physical admissibility of the coordinates. With respect to the physical system of reference S associated to the coordinates (x^i) (S being formed by the ∞^3 ideal particles having the x^4-lines as world lines with unit tangent vectors γ ($\gamma^4 = 1/\sqrt{-g_{44}}$, $\gamma^\alpha = 0$) the 4-vector $w^i(L)$ defining the direction of the vortices satisfies the condition

$$\gamma^i w_i(L) = \gamma^4 w_4(L) = 0 \quad \text{or} \quad w_4(L) = 0, \tag{7}$$

i.e. the tensorial index i of w_i is purely spatial.

Proceeding exactly as in paper I, we first express u_i and S_{ij} in Eq.(5) in terms of the corresponding natural projections $P_\theta(u_i) = -\gamma_i\gamma_j u^j$, $P_\Sigma(u_i) = (g_{ij}+\gamma_i\gamma_j)u^j = \gamma_{ij}u^j$, belonging respectively to the subspaces of the tangent space T_x at the event x, θ_x and Σ_x respectively parallel and orthogonal to γ, and of the projections $P_{\Sigma\Sigma}(S_{ij})$, $P_{\Sigma\theta}(S_{ij})$ and $P_{\theta\Sigma}(S_{ij})$ belonging respectively to the subspaces $\Sigma\otimes\Sigma$, $\Sigma\otimes\theta$, $\theta\otimes\Sigma$ of $T_x\otimes T_x$. With reference to (I-2.3, 2.5, 2.6) Eq.(5) may be written in the form

$$\nabla_{[i}\, h\widetilde{u}_{j]} + \nabla_{[i}\, h\tau_{j]} = \widetilde{S}_{ij} - \widetilde{S}_i\gamma_j + \gamma_i\widetilde{S}_j, \tag{8}$$

where

$$\widetilde{u}_j = P_\Sigma(u^j) = \Gamma\widetilde{v}_j, \quad \tau_j = P_\theta(u^j) = c\Gamma\gamma_j, \quad \left(\Gamma = \left(1-\widetilde{v}^2/c^2\right)^{-1/2}\right), \tag{9}$$

$\widetilde{v}_j$ being the standard relative 3-velocity as defined by (I-2.11) with $\widetilde{v}^2 = \gamma_{\alpha\beta}\widetilde{v}^\alpha\widetilde{v}^\beta$ its spatial form (I-2.14).

Using the expressions of the natural projections $P_{\Sigma\Sigma}$ and $P_{\Sigma\theta}+P_{\theta\Sigma}$ of the alternated derivatives $\nabla_{[i}\, h\widetilde{u}_{j]}$ (respectively $\nabla_{[i}\, h\tau_{j]}$) as given by (I-3.5) the corresponding projections of Eq.(8) may be exhibited in the form:

$$\text{Projection } \Sigma\Sigma: \quad (\widetilde{rot}\, h\widetilde{u})_{ij} + ch\Gamma\widetilde{\Omega}_{ij} = \widetilde{S}_{ij}, \tag{10a}$$

$$\begin{aligned}\text{Projection } \Sigma\theta+\theta\Sigma: \quad & \gamma_j\left\{\overline{\partial}_4\, h\widetilde{u}_i + \widetilde{\partial}_i\, c\Gamma h - c\Gamma h\left[\partial_4(\gamma_i/\gamma_4) - \widetilde{\partial}_i \log\sqrt{-g_{44}}\right]\right\} - \\ & -\gamma_i\left\{\overline{\partial}_4\, h\widetilde{u}_j + \widetilde{\partial}_j\, c\Gamma h - c\Gamma h\left[\partial_4(\gamma_j/\gamma_4) - \widetilde{\partial}_j \log\sqrt{-g_{44}}\right]\right\} = -\widetilde{S}_i\gamma_j + \gamma_i\widetilde{S}_j,\end{aligned} \tag{10b}$$

where $\overline{\partial}_4 = \gamma^4\partial_4$ and $\widetilde{\partial}_i = \partial_i - (\gamma_i/\gamma_4)\partial_4$ are respectively, in Cattaneo's terminology, the longitudinal and partial transverse derivatives. $\widetilde{\Omega}_{ij} = P_{\Sigma\Sigma}\,\Omega_{ij} = \gamma_4\left(\widetilde{\partial}_i(\gamma_j/\gamma_4) - \widetilde{\partial}_j(\gamma_i/\gamma_4)\right)$ is the space vortex tensor characterizing the system of reference S [5], [I-3.4]. We recall that in Eqs.(10a,b) the latin indices are purely spatial (i.e. the components $T_{ijk\ldots}$ vanish whenever a covariant index takes the value 4).

On multiplying Eq.(10b) by γ_j we obtain

$$\overline{\partial}_4(h\widetilde{u}_\alpha) + \widetilde{\partial}_\alpha\, c\Gamma h - c\Gamma h\left[\partial_4(\gamma_\alpha/\gamma_4) - \widetilde{\partial}_\alpha \log\sqrt{-g_{44}}\right] = -\widetilde{S}_\alpha. \tag{11}$$

Expressing $\widetilde{u}_\alpha$ in terms of the covariant components $\widetilde{v}_\alpha = \gamma_{\alpha\beta}\widetilde{v}^\beta = \gamma_{\alpha\beta}\, dx^\beta/dT$ of the standard relative 3-velocity $\widetilde{v}$, $dT = -\gamma_i\, dx^i/c$ being the interval of standard time relative to the system of reference S [5], [I-2.10, 2.14] and defining the standard relative specific enthalpy in the following manner

$$\bar{h} = h\Gamma \tag{12}$$

(the above definition is similar to that of the standard relative mass as introduced by Cattaneo [5], [I-2.15]; this is not surprising since $\bar{h}$ plays the role of an effective mass), Eqs.(10a) and (11) may be written

$$\left(\widetilde{rot}\,\bar{h}\,\tilde{v}\right)_{\alpha\beta} + c\bar{h}\,\widetilde{\Omega}_{\alpha\beta} = \widetilde{S}_{\alpha\beta}\,, \tag{13}$$

$$\bar{\partial}_4\left(h\,\tilde{v}_\alpha\right) = -c\bar{h}\,\tilde{\partial}_\alpha \log\sqrt{-g_{44}} - c\bar{h}\,\partial_4\left(\gamma_\alpha/\gamma_4\right) - c\,\tilde{\partial}_\alpha\,\bar{h} - \widetilde{S}_\alpha\,. \tag{14}$$

With reference to Eqs.(2) $\widetilde{S}_\alpha \left(= -\gamma_\alpha^r \gamma^s S_{rs}\right)$, in terms of the standard relative 3-velocity $\tilde{v}^\beta(L)$ and direction $w^\gamma(L)$ of the vortex, is given by:

$$\widetilde{S}_\alpha = -\frac{w}{2c}\tilde{\eta}_{\alpha\beta\gamma}\Gamma(L)\tilde{v}^\beta(L)w^\gamma(L) \quad \left(\Gamma(L) = \left(1 - \tilde{v}^2(L)/c^2\right)^{-1/2}\right) \tag{14a}$$

where $\tilde{\eta}_{ijl} = \eta_{ijlm}\gamma^m$.

Eq.(13) may be exhibited in yet a second form which is more convenient for physical interpretation. To this end, multiplying Eq.(13) by $\tilde{\eta}^{\alpha\rho\sigma}/2$ we introduce the dual vectors of the skew-symmetric tensors $\left(\widetilde{rot}\,\tilde{u}\right)_{ij}$ and $\widetilde{S}_{ij}$ [I-3.10, 3.12]

$$\left(\widetilde{rot}\,\tilde{u}\right)^\alpha = \frac{1}{2}\tilde{\eta}^{\alpha\rho\sigma}\left(\widetilde{rot}\,\tilde{u}\right)_{\rho\sigma}\,, \tag{15a}$$

$${}^*\widetilde{S}^\alpha = \frac{1}{2}\tilde{\eta}^{\alpha\rho\sigma}\,\widetilde{S}_{\rho\sigma} = \frac{w}{2c}\left(1 + \frac{\gamma_\sigma \tilde{v}^\sigma(L)}{c}\right)w^\alpha(L)\Gamma(L), \tag{15b}$$

so that Eq.(13) reads

$$\left(\widetilde{rot}\,\bar{h}\,\tilde{v}\right)^\alpha = -2\bar{h}\,\omega^\alpha + \frac{w}{2c}\left(1 + \frac{\gamma_\sigma \tilde{v}^\sigma(L)}{c}\right)\Gamma(L)w^\alpha(L) \tag{16}$$

where $\omega^\alpha\left(= \frac{c}{4}\tilde{\eta}^{\alpha\rho\sigma}\widetilde{\Omega}_{\rho\sigma}\right)$ is the angular velocity of the system of reference S [5], [I-3.14]. Eqs.(14) and (16) provide a description of the dynamics of vortices and of the velocity field of the superfluid in terms of standard quantities relative to the system of reference S associated to the physically admissible coordinates $\left\{x^i\right\}$.

The left hand side of Eq.(14) is the time derivative of the relativistic standard 3-momentum of a "superfluid particle"; accordingly on the right hand side appears the sum of the various forces acting on that "particle". The first two terms represent the influence of the gravitational field, the third term represents the force arising from the inhomogeneities of the velocity and of pressure fields, finally the last term expresses the force exerted by the quantum vortices on the superfluid (this force with a minus sign coincides with the Magnus force acting on the vortices).

As it is well known from Landau condition, in the absence of a gravitational field and of vortices, the curl of the momentum field must be equal to zero. In the case of a rotating superfluid the generation of vortices gives a nonzero contribution to the curl. Eq.(16) shows that the gravitational field gives the additional term $-2\bar{h}\,\omega^{\alpha}$ and the inhomogeneous pressure field modifies the term representing the contribution of vortices into ${}^{*}\widetilde{S}^{\alpha}$. Finally let us note that Eqs.(14) and (16) are valid for relativistic velocity fields within the frame of General relativity theory.

It is not without interest to obtain the Newtonian limits of Eqs.(14) and (16) in the absence of vortices. The derivation of the corresponding Newtonian equations from the standard relative equations (14) and (16) shows that Cattaneo's approach provides a simple method for the investigation and interpretation of physical phenomena in curved space-time.

To recover the Newtonian theory we have to make the following approximations: (i) static universe i.e. $g_{\alpha 4}=0$ which implies $\widetilde{\partial}_{\alpha}=\partial_{\alpha}$ and the vanishing of the space vortex tensor $\widetilde{\Omega}_{\alpha\beta}$ (or $\omega^{\alpha}=0$), (ii) $\partial_{\alpha}\log\sqrt{-g_{44}}=\partial_{\alpha}\left(1+\frac{2\varphi}{c^2}\right)^{1/2}\approx\partial_{\alpha}\left(\varphi/c^2\right)$, (iii) $\widetilde{\upsilon}_{\alpha}=\upsilon^{\alpha}=\frac{dx^{\alpha}}{dt}$, $\widetilde{\upsilon}^{4}=0$ and $\Gamma=\left(1-\widetilde{\upsilon}^2/c^2\right)^{1/2}\approx 1$.

Taking into account the above approximations Eq.(14) may be written in the form

$$\partial_4\upsilon^{\alpha}+c\,\partial_{\alpha}\left(\frac{\varphi}{c^2}+\partial_{\alpha}\frac{\upsilon^2}{2c^2}\right)+c\,\partial_{\alpha}\log h=0\,. \tag{17}$$

Setting $h=mc^2\left(1+\frac{h_N}{c^2}\right)$, where h_N is the enthalpy per particle in the Newtonian theory we get the well known equation of motion e.g. [6]

$$\frac{\partial\vec{\upsilon}}{\partial t}+\vec{\nabla}\left(\varphi+\frac{\upsilon^2}{2}+h_N\right)=0\,. \tag{18}$$

In the limiting situation under consideration Eq.(16) reduces to the Landau condition $\mathrm{rot}\vec{\upsilon}=0$ characterizing the superfluid velocity field.

3. *Conclusion*. Adopting Cattaneo's projection method we have presented an alternate formulation of the dynamical equations for superfluids valid in curved space-time. These equations display explicitly the various forces acting on a "superfluid particle" and have a form similar to the classical dynamical equations. In the Newtonian limit we recover the well known classical equations describing the motion of superfluid matter in Galilean space-time.

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ДИНАМИЧЕСКИЕ УРАВНЕНИЯ СВЕРХТЕКУЧЕЙ ЖИДКОСТИ В ИСКРИВЛЕННОМ ПРОСТРАНСТВЕ-ВРЕМЕНИ И МЕТОД ПРОЕКТИРОВАНИЯ КАТТАНЕО

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Тензорные уравнения, описывающие динамику сверхтекучей жидкости в Общей теории относительности, получены в том же виде, что и классические уравнения, написанные в 3-векторной форме. Приведены выражения для различных сил, действующих на элемент сверхтекучей жидкости. В ньютоновском пределе эти уравнения переходят в классические уравнения движения сверхтекучей жидкости в плоском пространстве-времени.

Ключевые слова: *сверхтекучая жидкость:динамические уравнения - Общая теория относительности*

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